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Three-dimensional minimal CR submanifolds in S^6 satisfying Chen's equality

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Abstract

In this paper, we classify 3-dimensional minimal CR submanifolds M of the nearly Kähler 6-dimensional sphere which satisfy Chen's basic equality, i.e. $\delta_M(p) = 2$, where $\delta_M(p) = \tau(p) - \inf K(p)$, for every $p \in M$.

MSC: 53B25; 53C42

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1. Introduction

Submanifolds of the nearly Kähler 6-dimensional sphere $S^6(1)$ have been an active field of research for years and many authors have described a lot of interesting geometric properties of these submanifolds (see for example [7,10,12,18]).

Considering $\mathbb{R}^7$ as the imaginary Cayley numbers, it is possible to introduce a vector cross product $\times$ on $\mathbb{R}^7$, which in its turn induces an almost complex structure J on the standard unit sphere S^6 in $\mathbb{R}^7$ which is compatible with the standard metric. It was shown by Calabi and Gluck, see [4], that this structure, from a geometric viewpoint, is the best possible almost complex structure on $S^6(1)$. Details about this construction are recalled in Section 2.

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With respect to the almost complex structure J, in the study of submanifolds, it is natural to study submanifolds for which J maps the tangent space into the tangent space (and hence also the normal space into the normal space) and those for which J maps the tangent into the normal space. The first class are called almost complex submanifolds and it was shown by Gray in [14] that they have to be two-dimensional (complex one-dimensional).

The second class of submanifolds mentioned, which by its definition have to be 1-, 2- or 3-dimensional, are called totally real submanifolds. By definition every curve is totally real. Minimal totally real surfaces have been studied and characterized in [2] and [3]. A 3-dimensional totally real submanifold is called a Lagrangian submanifold. In that case the almost complex structure exchanges at each point the tangent and normal space. These submanifolds were first investigated by Ejiri, [12], who showed that a Lagrangian submanifold is always orientable and minimal.

Further, let M be a submanifold in the nearly Kähler 6-sphere $S^6(1)$. A subspace $V \subset T_pM$ is called totally real if $JV \subset T_p^{\perp}M$, where T_pM and $T_p^{\perp}M$ denote the tangent space and the normal space of M at p, respectively. A submanifold M of $S^6(1)$ is called a CR submanifold if there exists on M a differentiable holomorphic distribution $\mathcal{H}$ (i.e. $J\mathcal{H} = \mathcal{H}$) such that its orthogonal complement $\mathcal{H}^{\perp} \subset TM$ is a totally real distribution [1]. A CR submanifold is called proper if it is neither totally real (i.e. $\mathcal{H}^{\perp} = TM$) nor holomorphic (i.e. $\mathcal{H} = TM$).

On the other hand, for a Riemannian n-manifold M^n denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pM^n$, $p \in M^n$. For an orthonormal basis $e_1, \dots, e_n$ of the tangent space T_pM^n, the scalar curvature τ at p is defined by

$$\tau = \sum_{i<j} K(e_i \wedge e_j).$$

For each point $p \in M^n$, let $(\inf K)(p) = \inf\{K(\pi) :$ plane sections $\pi \subset T_pM^n\}$. Then $\inf K$ is a well-defined function on M^n. Let δ_M denote the difference between the scalar curvature and $\inf K$, i.e.,

$$\delta_M(p) = \tau(p) - \inf K(p).$$

It is obvious that δ_M is a well-defined Riemannian invariant which is trivial when $n \leq 2$ (cf. [5] for details).

For a submanifold M^n in a Riemannian manifold $R^m(c)$ of constant sectional curvature c, the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature H^2 was first established in [5]:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)c. \tag{1}$$

It is a natural and very interesting problem to study and to understand submanifolds which satisfy the equality case of this inequality, which is known as Chen's basic equality (see, for example [9]). For such submanifolds there is a canonical distribution defined by

$$\mathcal{D}(p) = \{X \in T_pM^n | (n-1)h(X, Y) = n \langle X, Y\rangle \vec{H}, \forall Y \in T_pM^n\},$$

where $\vec{H}$ is the mean curvature vector field and h is the second fundamental form of M^n in a Riemannian manifold $R^m(c)$ of constant sectional curvature c. If the dimension of $\mathcal{D}(p)$ is constant, it is shown in [5] that the distribution $\mathcal{D}$ is completely integrable.

For minimal submanifolds of $S^m(1)$, the inequality (1) gives an upper bound for δ_M and reduces to

$$\delta_M \leq \frac{1}{2}(n+1)(n-2).$$

In this paper we will assume that M is a 3-dimensional minimal CR-submanifold of $S^6(1)$ which satisfies Chen's basic equality, i.e. $\delta_M = 2$. In the case that M is Lagrangian a complete classification has been obtained in [10]. Therefore, we may restrict ourselves here to the case that M is a proper CR-submanifold. Such submanifolds satisfying Chen's basic equality have been previously studied in [17], where the main result states:

Theorem 1. *There exist no 3-dimensional proper CR-submanifold in $S^6(1)$ satisfying Chen's basic equality under the condition that $\mathcal{D}$ is totally real.*

The main theorem we prove is the following:

Main Theorem. *Let M be a 3-dimensional minimal CR-submanifold in S^6 satisfying Chen's basic equality. Then M is a totally real submanifold or locally M is congruent with the immersion*

$$\begin{aligned} f(t,u,v) = {} & (\cos t\cos u\cos v, \sin t, \cos t\sin u\cos v, \cos t\cos u\sin v, 0, \\ & -\cos t\sin u\sin v, 0). \end{aligned} \tag{2}$$

Remark 1. We notice that (2) can also be described algebraically by the equations

$$x_5 = 0 = x_7, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_6^2 = 1, \quad x_3x_4 + x_1x_6 = 0,$$

from which we see that it can be seen as a hypersurface lying in a totally geodesic $S^4(1)$.

Remark 2. The first example of a 3-dimensional proper CR-submanifold was given by Sekigawa in [18]. Further examples, including a generalization of Sekigawa's example and a classification of those which are an orbit of a 3-dimensional simple subgroup of G_2, are given in [16].

Remark 3. It is proved in [6] that a submanifold of complex projective space satisfies Chen's basic equality if and only if the submanifold is a totally geodesic complex submanifold. Proper CR-submanifolds of complex hyperbolic spaces which satisfy the basic equality have been completely classified in [8].

2. Preliminaries

2.1. The vector cross product and the almost complex structure on $S^6(1)$

We give a brief exposition of how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from the Cayley multiplication. For further details about the Cayley numbers and their automorphism group G_2, we refer the reader to [19] and [15].

The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product $\times$ on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula

$$u \times v = \frac{1}{2}(uv - vu), \tag{3}$$

while the standard inner product on $\mathbb{R}^7$ is given by

$$(u, v) = -\frac{1}{2}(uv + vu). \tag{4}$$

It is now elementary [15] to show that

$$u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - (w, v)u, \tag{5}$$

and that the triple scalar product $(u \times v, w)$ is skew symmetric in u, v, w.

Conversely, Cayley multiplication on $\mathcal{O}$ is given in terms of the vector cross product and the inner product by

$$(r + u)(s + v) = rs - (u, v) + rv + su + (u \times v), \quad r, s \in Re(\mathcal{O}), u, v \in \mathrm{Im}(\mathcal{O}). \tag{6}$$

In view of (3), (4) and (6), it is clear that the group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\mathbb{R}^7$ preserving the vector cross product.

An ordered orthonormal basis $e_1, \ldots, e_7$ is said to be a G_2-frame if

$$e_3 = e_1 \times e_2, \quad e_5 = e_1 \times e_4, \quad e_6 = e_2 \times e_4, \quad e_7 = e_3 \times e_4. \tag{7}$$

For example, the standard basis $e_1, \ldots, e_7$ of $\mathbb{R}^7$ is a G_2-frame. Moreover, if e_1, e_2, e_4 are mutually orthogonal unit vectors with e_4 orthogonal to $e_1 \times e_2$, then e_1, e_2, e_4 determine a unique G_2-frame $e_1, \ldots, e_7$ and $(\mathbb{R}^7, \times)$ is generated by e_1, e_2, e_4 subject to the relations:

$$e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik}e_j - \delta_{ij}e_k - \delta_{jk}e_i. \tag{8}$$

Therefore, for any G_2-frame, we have the following very useful multiplication table [19]:

x	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

The standard nearly Kähler structure on $S^6(1)$ is then obtained as follows:

$$Ju = x \times u, \quad u \in T_x S^6(1), \quad x \in S^6(1).$$

It is clear that J is an almost complex structure on $S^6(1)$. In fact J is a nearly Kähler structure in the sense that the (2, 1)-tensor field G on $S^6(1)$ defined by

$$G(X, Y) = (\tilde{\nabla}_X J)Y,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$, is skew-symmetric. A straightforward computation also shows that

$$G(X, Y) = X \times Y - \langle x \times X, Y \rangle x.$$

For more information on the properties of $\cdot$, J and G, we refer to [2] and [11].

2.2. Minimal submanifolds of a nearly Kähler sphere $S^6(1)$ and Chen's basic equality

Let M be a Riemannian submanifold of a nearly Kähler sphere $S^6(1)$ and let us denote by $\tilde{\nabla}$ and ∇ the Riemannian connection of $S^6(1)$ and M, respectively, and by $\nabla^{\perp}$ the normal connection induced from $\tilde{\nabla}$ in the normal bundle $T^{\perp}M$ of M in $S^6(1)$. They are related by the following well-known Gauss equation

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{9}$$

for vector fields X, Y tangent to the submanifold, where h denotes the second fundamental form, and by the Weingarten equation

$$\tilde{\nabla}_X \xi = -A_{\xi} X + \nabla^{\perp}_X \xi \tag{10}$$

where A is the shape operator and ξ is a normal vector field.

It is well-known that the second fundamental form and the shape operator are related by $\langle h(X, Y), \xi \rangle = \left\langle A_{\xi} X, Y \right\rangle$. The mean curvature vector $\vec{H}$ of the immersion is given by

$$\vec{H} = \frac{1}{n} \text{trace } h.$$

A submanifold is said to be minimal if its mean curvature vector vanishes identically.

Denote by R the Riemann curvature tensor of M. Then the equation of Gauss is given by

$$\begin{aligned} R(X, Y; Z, W) = {} & \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle \\ & + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle, \end{aligned} \tag{11}$$

for vectors X, Y, Z, W tangent to M.

First, we recall the following result from [5], which we formulate here in an invariant way using the previously defined distribution $\mathcal{D}$, for 3-dimensional submanifolds of $S^6(1)$:

Lemma 1. *Let M be a 3-dimensional submanifold of a nearly Kähler sphere $S^6(1)$. Then*

$$\delta_M \leq \frac{9}{4} H^2 + 2. \tag{12}$$

Equality holds at a point p if and only if,

$$\mathcal{D}(p) = \{X \in T_p M^3 | 2h(X, Y) = 3 \langle X, Y \rangle \vec{H}, \forall Y \in T_p M^n\}, \tag{13}$$

has dimension greater or equal to 1.

We also need the following lemma from [17]:

Lemma 2. *Let M be a 3-dimensional minimal CR-submanifold in $S^6(1)$. If M satisfies Chen's basic equality and $\mathcal{D}$ is totally real, then M is a totally real submanifold.*

3. Proof of the main theorem

Since a 3-dimensional CR-submanifold is either totally real or proper, from now on we always assume that M is a 3-dimensional minimal proper CR-submanifold of $S^6(1)$ which satisfies Chen's equality at every point $p \in M$. In that case, the distribution $\mathcal{D}$ becomes the nullity distribution and we know that at each point p of M, the dimension of $\mathcal{D}(p)$ is at least 1. Then, we have the following lemma:

Lemma 3. *Let M be a 3-dimensional minimal submanifold of a nearly Kähler sphere $S^6(1)$ satisfying Chen's equality. Then, for any p in M,* $\dim \mathcal{D}(p) > 1$ *if and only p is a totally geodesic point. Moreover, in a neighborhood of a non-totally geodesic point, the distribution $\mathcal{D}$ is differentiable.*

Proof. If at a point p, $\dim \mathcal{D}(p) > 1$, as M is minimal, there exists an orthonormal basis $\{u_1, u_2, u_3\}$ at the point p such that $u_2, u_3 \in \mathcal{D}$. The definition of $\mathcal{D}$ then implies that

$$h(v, u_2) = h(v, u_3) = 0,$$

for any vector v. The minimality of M then implies that $h(u_1, u_1) = 0$. Hence p is a totally geodesic point.

Assume now that $\dim(\mathcal{D}) = 1$ in a neighborhood of p and that $\{u_1, u_2, u_3\}$ is an orthonormal basis at the point p such that u_3 spans $\mathcal{D}$. Then from the Gauss equation and the minimality of M it follows that

$$\begin{aligned}
&\mathrm{Ric}(u_1, u_1) = \mathrm{Ric}(u_2, u_2) = 1 - \|h(u_1, u_1)\|^2 - \|h(u_1, u_2)\|^2 < 1\\
&\mathrm{Ric}(u_1, u_2) = \mathrm{Ric}(u_1, u_3) = \mathrm{Ric}(u_2, u_3) = 0\\
&\mathrm{Ric}(u_3, u_3) = 1,
\end{aligned}$$

from which it follows that u_3 is the unique eigenvector of the Ricci tensor with eigenvalue 1. As the Ricci tensor is a differentiable operator, its eigenspaces with constant multiplicities are differentiable. □

Note that it is well known, see for example [13], that there does not exist a totally geodesic CR-submanifold of $S^6(1)$. Therefore the subset of non-totally geodesic points is an open dense subset of M. In the remainder we will restrict ourselves to this open dense subset. In that case, we denote by u_3 a differentiable unit vector field which spans $\mathcal{D}$. By the theorem of Sasahara, see Lemma 2 [17], restricting to an open dense subset if necessary, we know that $\mathcal{D}$ is not totally real. Therefore by projecting onto $\mathcal{H}$ and normalizing we can define a differentiable vector field E_1.

Since M is a 3-dimensional CR-submanifold in $S^6(1)$, it follows that we can choose orthonormal differentiable vector fields $\{E_1, E_2, E_3\} \in T_pM$ defined on a neighborhood of the point p, such that $\{E_1, E_2 = JE_1\} \in \mathcal{H}$, $E_3 \in \mathcal{H}^\perp$ and $\{JE_3, E_1 \times E_3, J(E_1 \times E_3)\} \in T_p^\perp M$.

We then introduce local functions $a_1, \ldots, c_3$ by

$$\begin{aligned}
&\nabla_{E_1} E_1 = a_1 E_2 + a_2 E_3, \quad \nabla_{E_1} E_2 = -a_1 E_1 + a_3 E_3, \quad \nabla_{E_1} E_3 = -a_2 E_1 - a_3 E_2,\\
&\nabla_{E_2} E_1 = b_1 E_2 + b_2 E_3, \quad \nabla_{E_2} E_2 = -b_1 E_1 + b_3 E_3, \quad \nabla_{E_2} E_3 = -b_2 E_1 - b_3 E_2,\\
&\nabla_{E_3} E_1 = c_1 E_2 + c_2 E_3, \quad \nabla_{E_3} E_2 = -c_1 E_1 + c_3 E_3, \quad \nabla_{E_3} E_3 = -c_2 E_1 - c_3 E_2.
\end{aligned}$$

By the construction of E_1, there exists a differentiable function θ such that

$$u_3 = \cos\theta E_1 + \sin\theta E_3.$$

We now define

$$\begin{aligned}
&u_2 = E_2, \quad u_1 = -\sin\theta E_1 + \cos\theta E_3, \quad u_4 = JE_3,\\
&u_5 = E_1 \times E_3, \quad u_6 = J(E_1 \times E_3).
\end{aligned}$$

Then, using Lemma 1, together with the minimality of M, we see that for M the shape operators $A_r = A_{u_r}$, $r = 4, 5, 6$ take the following forms:

$$A_r = \begin{pmatrix} \lambda_r & \mu_r & 0 \\ \mu_r & -\lambda_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

with respect to the tangent basis $\{u_1, u_2, u_3\}$.

Now, using (14), it follows that

$$\begin{aligned}
h(E_1, E_1) &= \lambda_1 \sin^2\theta JE_3 + \lambda_2 \sin^2\theta E_1 \times E_3 + \lambda_3 \sin^2\theta J(E_1 \times E_3),\\
h(E_1, E_2) &= -\mu_1 \sin\theta JE_3 - \mu_2 \sin\theta E_1 \times E_3 - \mu_3 \sin\theta J(E_1 \times E_3),\\
h(E_1, E_3) &= -\lambda_1 \sin\theta\cos\theta JE_3 - \lambda_2 \sin\theta\cos\theta E_1 \times E_3 - \lambda_3 \sin\theta\cos\theta J(E_1 \times E_3),\\
h(E_2, E_2) &= -\lambda_1 JE_3 - \lambda_2 E_1 \times E_3 - \lambda_3 J(E_1 \times E_3),\\
h(E_2, E_3) &= \mu_1 \cos\theta JE_3 + \mu_2 \cos\theta E_1 \times E_3 + \mu_3 \cos\theta J(E_1 \times E_3),\\
h(E_3, E_3) &= \lambda_1 \cos^2\theta JE_3 + \lambda_2 \cos^2\theta E_1 \times E_3 + \lambda_3 \cos^2\theta J(E_1 \times E_3).
\end{aligned} \tag{15}$$

Further, putting $e_1 = p$, $e_2 = E_1$, $e_3 = E_2 = JE_1$, $e_4 = E_3$, $e_5 = JE_3$, $e_6 = E_1 \times E_3$, $e_7 = -J(E_1 \times E_3)$, we conclude that this is at every point a G_2-frame and we can at every point apply the multiplication table given in Section 2.

Next, using the definition of the almost complex structure on $S^6(1)$, we have that

$$\begin{aligned}
D_X(E_2) &= D_X(JE_1) = D_X(p \times E_1) = X \times E_1 + p \times D_X E_1,\\
D_X(E_4) &= D_X(JE_3) = D_X(p \times E_3) = X \times E_3 + p \times D_X E_3,\\
D_X(E_1 \times E_3) &= D_X E_1 \times E_3 + E_1 \times D_X E_3,
\end{aligned}$$

where D is the standard connection of $\mathbb{R}^7$. Expressing the above equations using the formulae of Gauss (9) and Weingarten (10) together with the multiplication table, we obtain the following relations between the previously defined functions:

$$\begin{aligned}
&a_2 = -\mu_1 \sin\theta, \quad a_3 = -\lambda_1 \sin^2\theta,\\
&b_2 = -\lambda_1, \quad b_3 = \mu_1 \sin\theta,\\
&c_2 = \mu_1 \cos\theta, \quad c_3 = \lambda_1 \sin\theta\cos\theta,\\
&\lambda_2 = -\mu_3 \sin\theta, \quad \lambda_3 = \mu_2 \sin\theta,\\
&\lambda_2 \sin^2\theta = -\mu_3 \sin\theta, \quad \lambda_3 \sin^2\theta = \mu_2 \sin\theta,\\
&\mu_2 \cos\theta = -1 + \lambda_3 \sin\theta\cos\theta, \quad \mu_3 \cos\theta = -\lambda_2 \sin\theta\cos\theta
\end{aligned} \tag{16}$$

as well as the following expressions for the normal connection $\nabla^\perp$:

$$\begin{aligned}
\nabla^\perp_{E_1} JE_3 &= (1 + \lambda_3 \sin\theta\cos\theta) E_1 \times E_3 - \lambda_2 \sin\theta\cos\theta J(E_1 \times E_3),\\
\nabla^\perp_{E_2} JE_3 &= -\mu_3 \cos\theta E_1 \times E_3 + (\mu_2 \cos\theta - 1) J(E_1 \times E_3),\\
\nabla^\perp_{E_3} JE_3 &= -\lambda_3 \cos^2\theta E_1 \times E_3 + \lambda_2 \cos^2\theta J(E_1 \times E_3),\\
\nabla^\perp_{E_1} E_1 \times E_3 &= (-a_1 + \lambda_1 \sin\theta\cos\theta) J(E_1 \times E_3) - (\lambda_3 \sin\theta\cos\theta + 1) JE_3,\\
\nabla^\perp_{E_2} E_1 \times E_3 &= -(b_1 + \mu_1 \cos\theta) J(E_1 \times E_3) + \mu_3 \cos\theta JE_3,\\
\nabla^\perp_{E_3} E_1 \times E_3 &= -(c_1 + \lambda_1 \cos^2\theta) J(E_1 \times E_3) + \lambda_3 \cos^2\theta JE_3,
\end{aligned}$$

$$\nabla^{\perp}_{E_1} J(E_1 \times E_3) = \lambda_2 \sin\theta\cos\theta J E_3 + (a_1 - \lambda_1 \sin\theta\cos\theta) E_1 \times E_3,$$
$$\nabla^{\perp}_{E_2} J(E_1 \times E_3) = (1 - \mu_2\cos\theta) J E_3 + (b_1 + \mu_1\cos\theta) E_1 \times E_3,$$
$$\nabla^{\perp}_{E_3} J(E_1 \times E_3) = -\lambda_2 \cos^2\theta J E_3 + (c_1 + \lambda_1\cos^2\theta) E_1 \times E_3.$$

Using (16) it follows

$$\lambda_2(\sin^2\theta - 1) = 0, \quad \lambda_3(\sin^2\theta - 1) = 0. \tag{17}$$

First we interpret the above equations in the case when $\sin^2\theta = 1$, i.e. $\cos\theta = 0$. Then it follows that $u_3 = \pm E_3$, which implies that u_3 is a totally real subspace. Since $\mathcal{D} = \{u_3\}$, using Lemma 2 [17] a contradiction follows.

Now, we restrict ourselves to the case when $\sin^2\theta \neq 1$. Then it follows from (17) that $\lambda_2 = \lambda_3 = 0$. Further, from (16), it follows $\mu_2 = 0$ or $\sin\theta = 0$. Since $\mu_2 = 0$ gives a contradiction, it follows that $\sin\theta = 0$ and therefore, $a_2 = a_3 = b_3 = c_3 = \mu_3 = 0$, $b_2 = -\lambda_1$, $c_2 = \mu_1$, $\mu_2 = -1$. Now, by a straightforward computation, using (15), we obtain that there exist orthonormal vector fields $\{E_1, E_2, E_3\}$ defined on a neighborhood of the point p and differentiable functions λ_1, μ_1 such that:

$$\begin{aligned} &h(E_1, E_1) = h(E_1, E_2) = h(E_1, E_3) = 0\\ &h(E_2, E_2) = -\lambda_1 J E_3, \quad h(E_2, E_3) = \mu_1 J E_3 - E_1 \times E_3\\ &h(E_3, E_3) = \lambda_1 J E_3. \end{aligned} \tag{18}$$

Now, using the Codazzi equation which states that $(\nabla h)(X, Y, Z) = (\nabla_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$ is totally symmetric in X, Y and Z, we can prove the following

Lemma 4. *Let $\{E_1, E_2, E_3\}$ be the local orthogonal basis defined previously. Then $\lambda_1 = 0$ and the function μ_1 has to satisfy the following system of differential equations:*

$$E_1(\mu_1) = -1 - \mu_1^2, \quad E_2(\mu_1) = 0, \quad E_3(\mu_1) = 0. \tag{19}$$

Proof. For example, it follows from the Codazzi equation $(\nabla_{E_1} h)(E_2, E_2) = (\nabla_{E_2} h)(E_1, E_2)$ that

$$E_1(\lambda_1) = -\lambda_1(b_1 + \mu_1). \tag{20}$$

Further, it follows from the Codazzi equation $(\nabla_{E_1} h)(E_3, E_3) = (\nabla_{E_3} h)(E_1, E_3)$ that

$$E_1(\lambda_1) = -2\lambda_1\mu_1, \quad \lambda_1 = c_1. \tag{21}$$

Combining (20) and (21) it then follows that

$$\lambda_1 = 0 \quad \text{or} \quad b_1 = \mu_1. \tag{22}$$

The remaining equations follow similarly from the other Codazzi equations. □

Summarizing the above results, we see that we can express the components of the connection ∇ by

$$\begin{aligned} &\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0,\\ &\nabla_{E_2} E_1 = \mu_1 E_2, \quad \nabla_{E_2} E_2 = -\mu_1 E_1, \quad \nabla_{E_2} E_3 = 0,\\ &\nabla_{E_3} E_1 = \mu_1 E_3, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -\mu_1 E_1, \end{aligned} \tag{23}$$

and we obtain that the second fundamental form is given by $h(E_1, E_1) = h(E_1, E_2) = h(E_1, E_3) = h(E_2, E_2) = h(E_3, E_3) = 0$, $h(E_2, E_3) = \mu_1 JE_3 - E_1 \times E_3$. It is straightforward to compute that substituting these values into the Gauss equation or the Ricci equation does not yield any new conditions.

Finally, we are now ready to introduce coordinates and construct the corresponding CR-submanifold of $S^6(1)$. We look at the following system of differential equations for ρ:

$$E_1(\rho) = \rho\mu_1, \quad E_2(\rho) = 0, \quad E_3(\rho) = 0. \tag{24}$$

As $[E_1, E_2] = -\mu_1 E_2$, $[E_1, E_3] = -\mu_1 E_3$ and $[E_2, E_3] = 0$, it can be verified easily that all integrability conditions for ρ are satisfied.

Hence a solution to the above system (24) exists. As $[E_1, \rho E_2] = [\rho E_2, \rho E_3] = [E_1, \rho E_3] = 0$, it follows that there exist coordinates (t, u, v) in a neighborhood of $p \in M$, such that $\frac{\partial}{\partial t} = E_1$, $\frac{\partial}{\partial u} = \rho E_2$, $\frac{\partial}{\partial v} = \rho E_3$ and p corresponds to $(0, 0, 0)$. Using (19) and (24), it follows if necessary after translating the coordinate t that

$$\mu_1 = -\tan t, \quad \rho = \cos t. \tag{25}$$

Now, let $f : M \to S^6(1) \subset \mathbb{R}^7$ be a CR-immersion of a 3-dimensional minimal manifold satisfying Chen's basic equality. Rewriting now the formulae of Gauss (9) and Weingarten (10) using the coordinates (t, u, v), we obtain

$$f_{tt} = -f, \tag{26}$$
$$f_{tu} = \mu_1 f_u, \tag{27}$$
$$f_{tv} = \mu_1 f_v, \tag{28}$$
$$f_{uu} = -\mu_1\rho^2 f_t - \rho^2 f, \tag{29}$$
$$f_{uv} = \rho(\mu_1 f \times f_v - f_t \times f_v), \tag{30}$$
$$f_{vv} = -\rho^2(\mu_1 f_t + f). \tag{31}$$

Now, using (26), it follows

$$f(t, u, v) = A(u, v)\cos t + B(u, v)\sin t. \tag{32}$$

Substituting the expression (32) for f in (27) and (28), we obtain that B is a constant vector in $\mathbb{R}^7$. Using now (29), we get

$$A(u, v) = \cos u\, C(v) + \sin u\, D(v). \tag{33}$$

Repeating the similar procedure and using (30) and (31), we obtain $B \times D'(v) = C'(v)$, $B \times C'(v) = -D'(v)$ and $C''(v) = -C(v)$, $D''(v) = -D(v)$. Therefore, $C(v) = \cos v\, I + \sin v\, K$ and $D(v) = \cos v\, L + \sin v\, M$, where $I, K, L, M \in \mathbb{R}^7$ such that $B \times L = I$, $B \times M = K$, $B \times I = -L$ and $B \times K = -M$. Therefore, (32) and (33) imply that we can write

$$f(t, u, v) = B\sin t + \cos t\cos u(\cos vI + \sin vK) + \cos t\sin u(\cos vL + \sin vM). \tag{34}$$

Choosing the initial conditions $f(0, 0, 0) = u_1 = I$, $E_1(0, 0, 0) = u_2 = B$, $E_2(0, 0, 0) = u_3 = L$, $E_3(0, 0, 0) = u_4 = K$ and $M = -u_6$, we obtain Eq. (2).

On the other hand, using (2), a straightforward computation shows that the converse of the Main Theorem is obvious.

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